Application of Quasi-Linearization and Chebyshev Series to the Numerical Analysis of the Laminar Boundary-Layer Equations

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A method applicable to similar and nonsimilar flows is presented for computing the flowfield in an axisymmetric or two-dimensional, incompressible, laminar boundary layer. The equations of continuity and momentum are combined yielding a third order, nonlinear, parabolic, partial differential equation to be solved for the stream function. The streamwise derivatives are approximated by finite differences while the variation of the stream function and its higher derivatives normal to the flow are represented by Chebyshev series. At each streamwise location the solution to the nonlinear equation is obtained by iterating on a set of suitably linearized equations until a convergence criterion has been satisfied. Closed form series for Falkner-Skan profiles ranging from stagnation to separating flow, solutions to the Howarth retarded flow problem, and to the problem of flow over an elliptic cylinder are presented. The comparison of a limited amount of data indicates that the present method requires approximately the same number of arithmetic operations as a standard finite-difference technique for a given accuracy.

Nomenclature

a_r = coefficient of rth polynomial in Chebyshev series
f = dimensionless stream function
f' = dimensionless velocity parameter ($f' = u/u_e$)
F = dimensionless stream function appearing in Falkner-Skan equation
r = radius of body of revolution
R = radius parameter $(x/r^k)c/r^kc/x$
T_r = rth polynomial in Chebyshev series
u = component of velocity parallel to surface
v = component of velocity normal to surface
x = distance along surface measured from stagnation point or leading edge
Y = transformed y coordinate for similarity (Falkner-Skan) solutions
y = distance measured normal to surface
δ^* = displacement thickness
η = transformed y coordinate
τ = shear stress
θ = momentum thickness
μ = dynamic viscosity
ρ = density
ψ = stream function

1. Introduction

A METHOD for analyzing the incompressible laminar boundary-layer equations is presented. Certain features of a numerical method developed by Smith and co-workers[1-3] for solving these equations have been incorporated in the present investigation. In the aforementioned references, the partial differential equations governing the boundary layer are reduced to a set of ordinary differential equations by replacing the streamwise derivatives with finite-difference relations in accordance with ideas first proposed by Hartree and Womersley.[4] Reduction of the partial differential equations to ordinary equations has proved advantageous and this feature of the previous references has been retained in the present investigation. The primary advantage of it is that the vast literature available on ordinary differential equations can be brought to bear on the problem. Indeed this advantage has been utilized by employing certain techniques for treating nonlinear boundary-value problems that have recently received some attention.[5,6]

The nonlinear boundary-value problem to be considered was solved by a "shooting technique" in Refs. 1–3. That is, trial value boundary conditions were used at one boundary to provide sufficient information for solving the equation as an initial value problem. Various trial values were used and various solutions generated by integrating the equation until conditions at the other boundary are met with sufficient accuracy; for details see Refs. 1–3.

The shooting technique, although adequate for certain problems, possesses drawbacks. In some cases, the solutions are extremely sensitive to initial conditions and many trial solutions are needed to bracket the desired solution; the problem can be aggravated if small increments are needed for the streamwise finite-difference interval.[1] Moreover, because of the nature of the equation considered, in certain cases it is uncertain whether a given trial value is high or low and thus the logic associated with predicting successive trial values breaks down.

The aforementioned difficulties are in part due to the nonlinearity of the problem. Hence, employment of some type of linearization scheme was considered. The scheme adopted, termed quasi-linearization, utilizes a Taylor series expansion of the dependent variable about an approximate solution and the series is truncated to include only linear terms. As the problem is formulated here, this leads to a linear equation to be solved for the difference between the actual and approximate solutions, the coefficients of the equations to be solved being dependent on the approximate solution. The solution to this equation provides a new approximation; new coefficients are computed, and an iteration process is continued until some convergence criterion has been satisfied.

The solution to the linearized equation is represented by a series of Chebyshev polynomials.[7] The coefficients of the polynomials are related through a set of linear algebraic equations. Imposing the boundary conditions completes a set of equations necessary to solve for the coefficients which are obtained by Gauss elimination. It is of note that the employment of the series in a straightforward manner is made possible as a result of the quasi-linearization process. The Chebyshev series was adopted due to certain properties,

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discussed in Ref. 7, that make it economical from the standpoint of the mini-max principle.

The series has certain merits over a finite-difference solution in that it provides an analytic function for the dependent variable over the entire interval that is differentiable and integrable rather than a table of values given only as discrete mesh points. Moreover, if convergence is rapid the series will provide a more economical means of representing the solution than a finite-difference table, this consideration being of importance whenever problems of data storage become critical.

2. Analysis

The equations of continuity and momentum are considered for a two-dimensional or axisymmetric incompressible laminar boundary layer. These equations are combined through a dimensionless stream function f and written in a suitably transformed coordinate system as described in Ref. 1. The governing flowfield equation is

$$f''' + P(1 - f'^2) + Nff'' = x(f' \partial f'/\partial x - f''\partial f/\partial x) \quad (1)$$

where

$$\eta = (u_e/\nu x)^{1/2}\, y \quad (2)$$

$$\partial(\;)/\partial\eta \equiv (\;)' \quad (3)$$

$$f' = u/u_e \quad (4)$$

$$P = (x/u_e)\, du_e/dx \quad (5)$$

$$N = \frac{P+1}{2} + \frac{x}{r^k}\frac{dr^k}{dx}, \qquad \begin{Bmatrix} r = 0 \text{ two-dimensional} \\ r = 1 \text{ axisymmetric} \end{Bmatrix} \quad (6)$$

and the boundary conditions are

$$f(x,0) = f'(x,0) = 0 \quad (7)$$

$$f'(x,\eta_\infty) = 1 \quad (8)$$

The flowfield is divided into various stations (locations of fixed x) and at each station the streamwise derivatives appearing on the right-hand side of Eq. (1) are replaced with forward Lagrange difference approximations. The streamwise derivatives will then be

$$\partial f/\partial x = af + bf_{-1} + cf_{-2} \quad (9)$$

$$\partial f'/\partial x = cf' + bf'_{-1} + cf'_{-2} \quad (10)$$

where the subscripts -1 and -2 denote the two stations previous to x and the quantities a, b, and c depend on the chosen distribution of the independent variable x. At the zeroth station, $x = 0$, and the term containing streamwise derivatives in (1) vanishes at the next station, a two-point approximation must be used and the quantities a, b, and c are given by

$$a = (x - x_{-1})^{-1} \quad (11)$$

$$b = -(x - x_{-1})^{-1} \quad (12)$$

$$c = 0 \quad (13)$$

at succeeding stations a three-point difference approximation can be used and a, b, and c are given by

$$a = (x - x_{-1})^{-1} + (x - x_{-2})^{-1} \quad (14)$$

$$b = (x_{-2} - x)/(x - x_{-1})\,(x - x_{-2}) \quad (15)$$

$$c = (x - x_{-1})/(x - x_{-2})\,(x_{-1} - x_{-2}) \quad (16)$$

The flowfield can now be computed by solving a set of ordinary equations, successively, marching in the positive x direction.

2.1 Quasi-Linearization Technique

Upon substitution of the quantities given by Eqs. (9) and (10) into Eq. (1) and rearranging, the following third-order nonlinear equation is obtained

$$f''' + \alpha ff'' + \omega f'^2 + hf'' - h'f' + P = 0 \quad (17)$$

where

$$\alpha = (P + 1)/2 + R + \Delta_1 \quad (18)$$

$$\omega = -(P + \Delta_1) \quad (19)$$

$$h = \Delta_2 f_{-1} + \Delta_3 f_{-2} \quad (20)$$

$$h' = \Delta_2 f_{-1}' + \Delta_3 f'_{-2} \quad (21)$$

and

$$\Delta_1 = ax \quad (22)$$

$$\Delta_2 = bx \quad (23)$$

$$\Delta_4 = cx \quad (24)$$

Keeping in mind that solutions at previous stations (f_{-1}, f_{-2}, f_{-1}', f_{-2}') are known at a given station, Eq. (17) can be considered of the following form:

$$\varphi(f''',f'',f',f) = 0 \quad (25)$$

This equation can be linearized by expanding φ in a Taylor series about an approximate solution f_0 and truncating the series to include only linear terms. The resulting Taylor series is given by

$$\varphi = \varphi_0 + \left(\frac{\partial\varphi}{\partial f'''}\right)_0 (f''' - f'''_0) + \left(\frac{\partial\varphi}{\partial f''}\right)_0 (f'' - f''_0) + \left(\frac{\partial\varphi}{\partial f'}\right)_0 (f' - f'_0) + \left(\frac{\partial\varphi}{\partial f}\right)_0 (f - f_0) = 0 \quad (26)$$

where

$$\varphi_0 = \varphi(f''_0,f''_0,f'_0,f_0) \quad (27)$$

$$(\partial\varphi/\partial f''')_0 = 1 \quad (28)$$

$$(\partial\varphi/\partial f'')_0 = (\alpha f_0 + h) = B \quad (29)$$

$$(\partial\varphi/\partial f')_0 = 2\omega f'_0 = C \quad (30)$$

$$(\partial\varphi/\partial f)_0 = \alpha f_0 = D \quad (31)$$

The quantity ϵ, which is the difference between the exact and approximate solution, is now introduced

$$\epsilon^{(n)} = f^{(n)} - f_0^{(n)} \quad (32)$$

Substitution of the quantities given by Eqs. (27–32) into (26) yields the following ordinary linear equation:

$$\epsilon''' + B\epsilon'' + C\epsilon' + D\epsilon + \varphi_0 = 0 \quad (33)$$

The solution to Eq. (33) will make it possible to compute f and its higher derivatives from Eq. (32); these can be considered as a new approximation and the quantities B, C, D, and φ_0 can be re-evaluated and (39) solved again. The process is continued until a convergence criterion, to be discussed later, is satisfied. If the initial approximation f_0 is chosen so as to satisfy the boundary conditions (7) and (8) then the boundary conditions on Eq. (33) are

$$\epsilon(x,0) = \epsilon'(x,0) = 0 \quad (34)$$

$$\epsilon'(x,\eta_\infty) = 0 \quad (35)$$

2.2 Application of Chebyshev Series

Several properties of Chebyshev polynomials which will be of importance are presented. The rth polynomial in a

Chebyshev series is given by

$$T_r = \cos(r \arccos z) \quad (-1 \leq z \leq 1) \tag{36}$$

Using the weighting function $(1 - z^2)^{-1/2}$ the following orthogonality condition exists:

$$\int_{-1}^{1} \frac{T_m(z)\, T_n(z)}{(1 - z^2)^{1/2}}\, dz = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \neq 0 \\ \pi & m = n = 0 \end{cases} \tag{37}$$

Making use of this it can be shown that the coefficients a_r in an infinite series for $g(z)$ are given by

$$a_r = \left(\frac{2}{\pi}\right) \int_{-1}^{1} g(z)\, T_r(z)\, (1 - z^2)^{-1/2} dz \tag{38}$$

If $g(z)$ is not expressed analytically or if it is such that the integration is difficult the following approximation is useful:

$$a_r = \frac{2}{N} \sum_{j=0}^{N} g\left(\frac{\cos\pi j}{N}\right)\left(\frac{\cos\pi jr}{N}\right) \tag{39}$$

where the function $g(z)$ is now approximated by

$$g(z) = \frac{a_0}{2} + \sum_{r=1}^{N} a_r T_r(z) \tag{40}$$

From Eq. (36) it follows that

$$\frac{1}{r+1} T'_{r+1} - \frac{1}{r-1} T'_{r-1} = 2T_r \tag{41}$$

Expressing the derivative of g as a Chebyshev series with coefficients a'_r it can be shown from Eq. (41) that the following relationship exists between the coefficients of g and g':

$$a'_{r-1} = a'_{r+1} + 2ra_r \quad (1 \leq r \leq N) \tag{42}$$

and the quantities a'_N and a'_{N+1} vanish since from differentiation the order of the series for $g'(z)$ will be one less than the order of the series for $g(z)$. If the coefficients of g and g' are arranged in a column matrix Eq. (42) can be written

$$\begin{bmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \cdot \\ \cdot \\ \cdot \\ a'_{N-1} \end{bmatrix} = \begin{bmatrix} 2\ 0\ 6\ 0\ 10\ (1 - (-1)^N)N \\ \quad 4\ 0\ 8\ 0 \\ \qquad 6\ 0\ 10 \\ \\ \\ \\ \qquad\qquad\qquad 2N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{bmatrix} \tag{43}$$

Similar relations exist between a''_r and a'_r and a'''_r and a''_r. Since it will be necessary to express the product of two Chebyshev series itself as a Chebyshev series we will consider the product of two series f and g given by

$$f = \frac{b_0}{2} + \sum_{r=1}^{N} b_r T_r \tag{44}$$

$$g = \frac{a_0}{2} + \sum_{r=1}^{N} a_r T_r \tag{45}$$

The product can be shown to be given by

$$fg = \frac{a_0 b_0}{2} + \sum_{r=1}^{N}\left(\sum_{j=1}^{2N} a_j(b_{|j-r|} + b_{j+r})\right) T_r \tag{46}$$

where all coefficients with subscripts greater than N are taken to be zero. If the product is considered to be given by

$$fg = \frac{c_0}{2} + \sum_{r=1}^{N} c_r T_r \tag{47}$$

then the summation in parentheses in Eq. (46) gives the coefficients c_r. The solution to Eq. (33) is desired over the interval $(0, \eta_\infty)$ and the standard Chebyshev polynomial is defined over the interval $(-1, 1)$. This situation can be handled by either transforming the polynomials and their corresponding relations to the new interval or transforming the variable η. Since η_∞ may vary from one problem to the next, transformation of the polynomials would involve generating a new set for each problem; hence, the latter alternative was chosen and the following transformation is introduced:

$$z = (2/\eta_\infty)\, \eta - 1 \tag{48}$$

$$\epsilon'(\eta) = \epsilon'(z)\, dz/d\eta \tag{49}$$

Equation (33) becomes

$$\epsilon''' + \bar{B}\epsilon'' + \bar{C}\epsilon' + \bar{D}\epsilon + \bar{\varphi}_0 = 0 \tag{50}$$

where

$$\bar{B} = (\eta_\infty/2)\, B \tag{51}$$

$$\bar{C} = (\eta_\infty/2)^2\, C \tag{52}$$

$$\bar{D} = (\eta_\infty/2)^3\, D \tag{53}$$

$$\bar{\varphi}_0 = (\eta_\infty/2)^3\, \varphi_0 \tag{54}$$

and the boundary conditions are now

$$\epsilon(x,-1) = \epsilon'(x,-1) = 0 \tag{55}$$

$$\epsilon'(x,1) = 0 \tag{56}$$

The following solution to Eq. (50) is assumed

$$\epsilon = \frac{a_0}{2} + \sum_{r=1}^{N} a_r T_r \tag{57}$$

$$\epsilon' = \frac{a'_0}{2} + \sum_{r=1}^{N-1} a'_r T_r \tag{58}$$

$$\epsilon'' = \frac{a''_0}{2} + \sum_{r=1}^{N-2} a''_r T_r \tag{59}$$

$$\epsilon''' = \frac{a'''_0}{2} + \sum_{r=1}^{N-3} a'''_r T_r \tag{60}$$

The coefficients $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{\varphi}_0$ are known either from the approximate solution given initially or from successive approximations obtained by solving Eq. (50) and are also expressed as Chebyshev series.

At any given station in the flowfield the initial values of $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{\varphi}_0$ are obtained from the final solution at the previous station; at the zeroth station these quantities are obtained using a polynomial for the stream function satisfying the following conditions as an initial guess

$$f(0) = f'(0) = f''(\eta_\infty) = f'''(\eta_\infty) = f^{\text{iv}}(\eta_\infty) = 0$$

$$f'(\eta_\infty) = 1$$

The polynomial satisfying these conditions is

$$f(\eta) = \eta - (\eta_\infty/5)\, [1 - \{1 - (\eta/\eta_\infty)^5\}] \tag{61}$$

The series for $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{\varphi}_0$ which are known and the unknown series given by Eqs. (57–60) are substituted into Eq. (50); product relations of the form given by (46) are used to express the equation as a Chebyshev series of order N-3. The polynomials $T_r(z)$ over the interval $(-1, 1)$ are linearly independent and the equation will be satisfied if coefficients of like polynomials vanish. The system of equations will be

$$a'''_0 + a''_0\bar{b}_0 + a'_0\bar{c}_0 + a_0\bar{d}_0 + \bar{\varphi}_0 = 0 \tag{62a}$$

$$a'''_r + \sum_{j=1}^{2(N-3)} a''_j(\bar{b}_{|j-r|} + \bar{b}_{j+r}) + \sum_{j=1}^{2(N-3)} a'_j(\bar{c}_{|j-r|} + \bar{c}_{j+r}) + \sum_{j=1}^{2(N-3)} a_j(\bar{d}_{|j-r|} + \bar{d}_{j+r}) + \bar{\varphi}_r = 0 \tag{62b}$$

Table 1 Chebyshev coefficients c_r for stream function $F(z)$; $\beta = 1.0$, $\eta_\infty = 5.0$, $\delta = 10^{-7}$

	N				
r	6	8	10	12	14
0	3.950988	3.959255	3.959122	3.959139	3.959138
1	2.271418	2.273128	2.273071	2.273078	2.273078
2	1.586652×10^{-1}	1.593772×10^{-1}	1.594116×10^{-1}	1.594112×10^{-1}	1.594111×10^{-1}
3	-9.114729×10^{-2}	-8.711225×10^{-2}	-8.715557×10^{-2}	-8.715339×10^{-2}	-8.715348×10^{-2}
4	3.775569×10^{-2}	3.583515×10^{-2}	3.585731×10^{-2}	3.585688×10^{-2}	3.585689×10^{-2}
5	-8.043701×10^{-3}	-1.034490×10^{-2}	-1.017599×10^{-2}	-1.017377×10^{-2}	-1.017385×10^{-2}
6	3.124542×10^{-4}	1.411643×10^{-3}	1.397970×10^{-3}	1.396685×10^{-3}	1.396696×10^{-3}
7		3.980537×10^{-4}	2.767144×10^{-4}	2.679061×10^{-4}	2.677199×10^{-4}
8		-1.826638×10^{-4}	-1.876894×10^{-4}	-1.864332×10^{-4}	-1.863102×10^{-4}
9			2.678513×10^{-5}	3.350352×10^{-5}	3.415024×10^{-5}
10			3.218718×10^{-6}	3.072690×10^{-6}	2.902941×10^{-6}
11				-1.605884×10^{-6}	-2.119193×10^{-6}
12				-7.642309×10^{-8}	1.047558×10^{-9}
13					1.291435×10^{-7}
14					-1.175728×10^{-8}

for

$$(1 \leq r \leq N - 3)$$

where $\bar{b}_r, \bar{c}_r, \bar{d}_r$, and $\bar{\varphi}_r$ are the coefficients of the series for $\bar{B}$, $\bar{C}, \bar{D}$, and $\bar{\varphi}_0$, respectively and are known quantities. Furthermore, all coefficients with subscripts greater than N in the preceding summations vanish. There are a total of $4N$-2 unknown coefficients appearing in Eqs. (57–60) and a total of N-2 equations in the set given by (62). In addition, there are N equations of the form (42) relating a'_r and a_r, N-1 of the same form relating a''_r and a'_r, and N-2 relating a'''_r and a''_r. The three boundary conditions complete the set necessary for solving the system and are given by

$$\epsilon(-1) = a_0/2 - a_1 + a_2 - \ldots (-1)^N a_N = 0 \quad (63)$$

$$\epsilon'(1) = a'_0/2 + a'_1 + a'_2 + \ldots + a'_N = 0 \quad (64)$$

$$\epsilon'(-1) = a'_0/2 - a'_1 + a'_2 - \ldots (-1)^N a'_N = 0 \quad (65)$$

The solution to the nonlinear equation (17) is obtained as the solution to the linear equation (33) for ϵ goes to zero after successive iterations. Since the quantity ϵ may oscillate about the desired solution the following convergence criterion is considered:

$$\frac{1}{\eta_\infty}\left(\int_0^{\eta_\infty} \epsilon^2(\eta)\, d\eta\right)^{1/2} < \delta \quad (66)$$

Over the interval $(-1, 1)$ this criterion becomes

$$\int_{-1}^{1} \epsilon^2(z)\, dz < 2\delta^2 \quad (67)$$

Introducing the weighting function from Eq. (37) it can be seen that the above condition will be satisfied if

$$\int_{-1}^{1} \frac{\epsilon^2(z)}{(1 - z^2)^{1/2}}\, dz < 2\,\delta^2 \quad (68)$$

and finally if $\epsilon(z)$ is expressed as a Chebyshev series this inequality will hold if

$$\sum_{r=0}^{N} |a_r| < \left(\frac{4}{\pi}\right)^{1/2} \delta \quad (69)$$

This inequality was used to establish convergence in the present study where δ is chosen according to the accuracy requirements on the solution to any given problem.

The linear algebraic equations to be solved for the coefficients of the series for ϵ''', ϵ'', ϵ', and ϵ can be solved by various standard techniques. A method was chosen so as to provide the coefficients of ϵ only, the other coefficients are then constructed from successive applications of the operator matrix in (43). Standard gauss elimination proved suitable for this since it permits a subset of m unknowns to be obtained from any set of n equations with n unknown where $m \leq n$. Single floating point arithmetic on a computer with 48-bit words capable of storing numbers in the range $2^{-128} \leq f \leq 2^{127}$ to 39-bit accuracy was used. No ill-conditioning was found.

3. Results and Discussion

In this section, solutions to various problems are presented and comparisons are shown. Convergence and accuracy considerations are discussed.

Table 2 Chebyshev coefficients c_r for stream function $F(z)$; $\beta = 0.2$, $\eta_\infty = 6.0$, $\delta = 10^{-7}$

	N				
r	6	8	10	12	14
0	4.438233	4.452581	4.453522	4.453508	4.453319
1	2.624019	2.630256	2.630461	2.630393	2.630397
2	2.465655×10^{-1}	2.466501×10^{-1}	2.462435×10^{-1}	2.462708×10^{-1}	2.462702×10^{-1}
3	-1.230612×10^{-1}	-1.198426×10^{-1}	-1.196228×10^{-1}	-1.196434×10^{-1}	-1.196430×10^{-1}
4	3.662402×10^{-2}	3.752744×10^{-2}	3.742528×10^{-2}	3.743375×10^{-2}	3.743359×10^{-2}
5	-6.587331×10^{-4}	-2.994510×10^{-3}	-3.723429×10^{-3}	-3.741142×10^{-3}	-3.740492×10^{-3}
6	-2.006845×10^{-3}	-2.777741×10^{-3}	-2.578637×10^{-3}	-2.560975×10^{-3}	-2.561586×10^{-3}
7		4.732516×10^{-4}	9.919214×10^{-4}	1.067377×10^{-3}	1.069055×10^{-3}
8		2.024918×10^{-4}	1.357639×10^{-4}	9.837383×10^{-5}	9.629822×10^{-5}
9			-1.157324×10^{-4}	-1.735652×10^{-4}	-1.790987×10^{-4}
10			3.636587×10^{-6}	2.426558×10^{-5}	2.926196×10^{-5}
11				1.391430×10^{-5}	1.825606×10^{-5}
12				-3.822067×10^{-6}	-7.277768×10^{-6}
13					-1.088509×10^{-6}
14					8.030951×10^{-7}

Table 3 Chebyshev coefficients c_r for stream function $F(z)$; $\beta = 0.0$, $\eta_\infty = 6.0$, $\delta = 10^{-7}$

	N				
r	6	8	10	12	14
0	4.138117	4.135320	4.135201	4.134866	4.134921
1	2.512022	2.512560	2.512509	2.512365	2.512392
2	3.030663×10^{-1}	3.022346×10^{-1}	3.021864×10^{-1}	3.022083×10^{-1}	3.022084×10^{-1}
3	-1.234321×10^{-1}	-1.254391×10^{-1}	-1.253923×10^{-1}	-1.254076×10^{-1}	-1.254079×10^{-1}
4	2.236862×10^{-2}	2.554563×10^{-2}	2.556980×10^{-2}	2.557102×10^{-2}	2.557181×10^{-2}
5	3.954671×10^{-3}	4.854416×10^{-3}	4.492560×10^{-3}	4.474108×10^{-3}	4.475760×10^{-3}
6	-1.948976×10^{-3}	-3.949062×10^{-3}	-4.141780×10^{-3}	-4.132954×10^{-3}	-4.133758×10^{-3}
7		-1.014004×10^{-4}	1.885530×10^{-4}	2.795662×10^{-4}	2.837029×10^{-4}
8		3.827798×10^{-4}	5.608193×10^{-4}	5.615968×10^{-4}	5.591928×10^{-4}
9			-6.828784×10^{-5}	-1.406654×10^{-4}	-1.558741×10^{-4}
10			-4.650521×10^{-5}	-5.711340×10^{-5}	-5.440832×10^{-5}
11				1.773669×10^{-5}	3.011770×10^{-5}
12				4.068268×10^{-6}	3.486120×10^{-6}
13					-3.166177×10^{-6}
14					-8.540984×10^{-8}

3.1 Similar Flows

For flows where the external velocity is given by

$$u_e \alpha x^{\beta/(2-\beta)}$$

it can be shown that the stream function can be written in terms of a similarity variable

$$Y = [(2 - \beta)^{-1/2} (u_e/\nu x)^{1/2} y]$$

The right-hand side of Eq. (1) vanishes, the quantity P is replaced with β, N with unity, resulting in the well known Falkner-Skan equation for the stream function $F(Y)$ which is related to $f(\eta)$ through

$$F(Y) = (2 - \beta)^{1/2} f(\eta)$$

and

$$\partial/\partial Y = (2 - \beta)^{1/2}\, \partial/\partial\eta$$

Solutions to the Falkner-Skan equation for various values of the parameter β were obtained and compared to the highly accurate results given in Ref. 8. The value of $F''(0)$ in this reference is given to six places and the solutions are an improvement over those obtained earlier by Hartree. Computed values of Chebyshev coefficients c_r for the stream function $F(z)$ over the interval $(-1, 1)$ given by

$$F(z) = \frac{c_0}{2} + \sum_{r=1}^{N} c_r T_r(z)$$

are given for values of $\beta = 1, 0.2, 0.0, -0.14$, and -0.198838 in Tables 1–5. The stream function can be obtained over the interval $(0,\eta_\infty)$ using the transformation given by Eq. (58). Values of the shear parameter, $F''(0)$ or F''_w, and values of the displacement thickness parameter,

$$\delta^*\left(\frac{u_e}{\nu x}\right)^{1/2} = \int_0^{Y_\infty} (1 - F')\, dY$$

are compared in Table 6 with those given in Ref. 8. Tables 1–6 also exhibit the effect of N on accuracy. The number of iterations L needed to satisfy the convergence criterion given by Eq. (79) also is demonstrated; the value of δ used in this equation was taken as 10^{-7}. The results indicate that over a wide range of β's a relatively small number of terms will insure sufficient accuracy for all practical purposes. However, no generalization can be made regarding the effect of the value of β on accuracy for a given value of N. For example, a seven term series ($N = 6$) gives higher accuracy for a flow with $\beta = 0.2$ than with $\beta = 0.0$; but a nine term series yields higher accuracy for a flow with $\beta =$

Table 4 Chebyshev coefficients c_r for stream function $F(z)$; $\beta = -0.14$, $\eta_\infty = 7.0$, $\delta = 10^{-7}$

	N				
r	8	12	16	18	20
0	4.579016	4.570405	4.570637	4.570647	4.570646
1	2.850014	2.846608	2.846719	2.846724	2.846724
2	3.998665×10^{-1}	4.011788×10^{-1}	4.011663×10^{-1}	4.011655×10^{-1}	4.011656×10^{-1}
3	-1.583323×10^{-1}	-1.585245×10^{-1}	-1.585283×10^{-1}	-1.585285×10^{-1}	-1.585285×10^{-1}
4	2.238444×10^{-2}	2.228777×10^{-2}	2.229369×10^{-2}	2.229401×10^{-2}	2.229397×10^{-2}
5	1.436630×10^{-2}	1.435011×10^{-2}	1.435512×10^{-2}	1.435504×10^{-2}	1.435505×10^{-2}
6	-7.232395×10^{-3}	-8.071394×10^{-3}	-8.075872×10^{-3}	-8.075869×10^{-3}	-8.075866×10^{-3}
7	-6.975754×10^{-4}	-2.994492×10^{-4}	-2.882766×10^{-4}	-2.882234×10^{-4}	-2.882290×10^{-4}
8	8.242076×10^{-4}	1.557228×10^{-3}	1.552645×10^{-3}	1.552677×10^{-3}	1.552673×10^{-3}
9		-2.384459×10^{-4}	-2.948576×10^{-4}	-2.947068×10^{-4}	-2.947078×10^{-4}
10		-2.379634×10^{-4}	-2.348088×10^{-4}	-2.348908×10^{-4}	-2.348956×10^{-4}
11		4.418998×10^{-5}	1.012725×10^{-4}	1.017829×10^{-4}	1.017628×10^{-4}
12		2.350332×10^{-5}	2.644960×10^{-5}	2.608224×10^{-5}	2.609394×10^{-5}
13			-2.170252×10^{-5}	-2.358491×10^{-5}	-2.366408×10^{-5}
14			-1.699970×10^{-6}	-1.036894×10^{-6}	-9.694332×10^{-7}
15			2.578214×10^{-6}	4.234154×10^{-6}	4.510718×10^{-6}
16			1.260449×10^{-8}	-3.656744×10^{-7}	-5.114731×10^{-7}
17				-4.582650×10^{-7}	-7.032261×10^{-7}
18				7.303988×10^{-8}	1.749301×10^{-7}
19					6.888496×10^{-8}
20					-2.443974×10^{-8}

Table 5 Chebyshev coefficients c_r for stream function $F(z)$; $\beta = -0.198838$, $\eta_\infty = 9.0$, $\delta = 10^{-7}$

	N	
r	25	30
0	5.428033	5.423859
1	3.520055	3.518383
2	6.056202×10^{-1}	6.063549×10^{-1}
3	-2.338924×10^{-1}	-2.338726×10^{-1}
4	1.776091×10^{-2}	1.751661×10^{-2}
5	3.839845×10^{-2}	3.851597×10^{-2}
6	-1.944370×10^{-2}	-1.941928×10^{-2}
7	-1.770301×10^{-3}	-1.815345×10^{-3}
8	5.156861×10^{-3}	5.163541×10^{-3}
9	-1.002091×10^{-3}	-9.894973×10^{-4}
10	-1.074072×10^{-3}	-1.079815×10^{-3}
11	5.349579×10^{-4}	5.324837×10^{-4}
12	1.560535×10^{-4}	1.584827×10^{-4}
13	-1.770577×10^{-4}	-1.768775×10^{-4}
14	-2.434579×10^{-6}	-3.213673×10^{-6}
15	4.685948×10^{-5}	4.697621×10^{-5}
16	-8.868850×10^{-6}	-8.664626×10^{-6}
17	-1.034406×10^{-5}	-1.042069×10^{-5}
18	4.270689×10^{-6}	4.225353×10^{-6}
19	1.814731×10^{-6}	1.846418×10^{-6}
20	-1.407208×10^{-6}	-1.410203×10^{-6}
21	-1.972630×10^{-7}	-1.997391×10^{-7}
22	3.485998×10^{-7}	3.871341×10^{-7}
23	-5.577607×10^{-9}	-2.244510×10^{-8}
24	-4.837090×10^{-8}	-9.085409×10^{-8}
25	4.456096×10^{-9}	2.214415×10^{-8}
26		1.803738×10^{-8}
27		-7.622229×10^{-9}
28		-2.868155×10^{-9}
29		1.243478×10^{-9}
30		2.899129×10^{-10}

-0.14 than with $\beta = 0.2$. Comparisons have shown that the velocity profiles are at least as accurate over the whole interval as values of $F''(0)$.

As β decreases, the boundary-layer thickens and the value of η_∞ must increase to insure an asymptotic solution at η_∞. The effect of η_∞ on the results, however, is negligible providing the solution is truly asymptotic. It was found that values of $F''(0)$ are not affected in the fifth place by the choice of η_∞ providing $F''(\eta_\infty)$ does not exceed 0.5×10^{-3}.

3.2 Howarth Retarded Flow

The solution to the Howarth retarded flow problem where the external velocity distribution is given by

$$u_e = 1 - x/8$$

was obtained. The velocity distribution given previously represents a highly decelerating flow leading to separation a short distance from the boundary-layer origin and thus presents a severe test on the capabilities of any numerical method.

The flow was studied using a value of 8.5 for η_∞; up to $x = 0.88$ twenty-five terms were used. The case was stopped at $x = 0.88$ and restarted using 35 terms until the final station. The value of δ used in the convergence criterion was taken as 10^{-8}.

Values for the parameter f''_w are shown and compared with the computed values of Smith and Clutter[1] and Hartree[9] in Table 7 for every third value of x; computed values of displacement thickness also are shown. The agreement is good except in the immediate vicinity of separation. The extrapolated values of where separation takes place (i.e., $f''_w \to 0$) agree within 1.25%. The present results agree better with Hartree's results than with Smith's. It is pointed out that finer spacing of the x stations in the region of separation and employment of double precision arithmetic could possibly yield better results. Smith and Clutter[1] reported that their method could not handle extremely small steps and consequently they were not able to march up to separation. This problem has not been encountered with the present method. However, further study is required to determine exactly what the limitations are of the present method in the vicinity of separation.

3.3 Comparison with Finite-Difference Method

Of particular interest is a comparison of the effort required to attain a solution of given accuracy between a standard finite-difference method and the present series solution.

Table 6 Computed values of shear parameter and displacement thickness

β	N	Y_∞	L	F''_w	F''_w (Ref. 8)	$\int_0^{Y_\infty}(1-F')dY$	$\int_0^{Y_\infty}(1-F')dY$ (Ref. 8)
2.0	20	4.0	6	1.68722	1.68722	4.97429×10^{-1}	0.49743
1.6	20	4.0	6	1.52151	1.52151	5.44017×10^{-1}	0.54402
1.2	20	4.0	6	1.33572	1.33572	6.06892×10^{-1}	0.60690
1.0	18	4.0	6	1.23259	1.23259	6.47895×10^{-1}	0.64790
1.0	18	9.0	6	1.23259	1.23259	6.47895×10^{-1}	0.64790
0.2	6	6.0	4	6.84280×10^{-1}	6.86708×10^{-1}	9.99402×10^{-1}	0.98416
0.2	8	6.0	4	6.88710×10^{-1}	6.86708×10^{-1}	9.84215×10^{-1}	0.98416
0.2	10	6.0	4	6.86487×10^{-1}	6.86708×10^{-1}	9.84018×10^{-1}	0.98416
0.2	12	6.0	4	6.86737×10^{-1}	6.86708×10^{-1}	9.84168×10^{-1}	0.98416
0.2	14	6.0	4	6.86706×10^{-1}	6.86708×10^{-1}	9.84159×10^{-1}	0.98416
0.0	6	6.0	4	4.83847×10^{-1}	4.69600×10^{-1}	1.21491	1.21678
0.0	8	6.0	4	4.69732×10^{-1}	4.69600×10^{-1}	1.21625	1.21678
0.0	10	6.0	4	4.69515×10^{-1}	4.69600×10^{-1}	1.21654	1.21678
0.0	12	6.0	4	4.69632×10^{-1}	4.69600×10^{-1}	1.21682	1.21678
0.0	14	6.0	4	4.69595×10^{-1}	4.69600×10^{-1}	1.21677	1.21678
0.0	16	6.0	4	4.69601×10^{-1}	4.69600×10^{-1}	1.21678	1.21678
−0.14	8	7.0	5	2.39507×10^{-1}	2.39736×10^{-1}	1.58930	1.59590
−0.14	12	7.0	6	2.39772×10^{-1}	2.39736×10^{-1}	1.59612	1.59590
−0.14	16	7.0	6	2.39741×10^{-1}	2.39736×10^{-1}	1.59591	1.59590
−0.14	18	7.0	6	2.39735×10^{-1}	2.39736×10^{-1}	1.59590	1.59590
−0.14	20	7.0	6	2.39736×10^{-1}	2.39736×10^{-1}	1.59590	1.59590
−0.14	18	9.0	6	2.39742×10^{-1}	2.39736×10^{-1}	1.59590	1.59590
−0.19	20	7.0	7	8.57071×10^{-2}	0.085700	2.00675	2.00676
−0.19	25	8.0	6	8.56998×10^{-2}	0.085700	2.00675	2.00676
−0.195	25	8.0	7	5.51719×10^{-2}	0.055172	2.11704	2.11705
s[a]	25	9.0	7	6.3903×10^{-4}	0.0	2.35563	2.35885
s[a]	30	9.0	7	1.4681×10^{-5}	0.0	2.35877	2.35885

[a] Separation profile $\beta_s = -0.198838$

Comparisons between the two methods for solutions to the Falkner-Skan equation are given. Reference 10 describes an implicit finite-difference scheme that has been applied to both incompressible laminar and turbulent flows; the comparisons will be presented for laminar flow. A 5-point Lagrange central-difference approximation is employed in Ref. 10 for the results used in the comparison. All terms in the dependent variable appearing in Eq. (1) were linearized in order to obtain a set of linear algebraic equations for the dependent variable upon application of the difference operator at each mesh point. With an internal size $\Delta\eta$ the order of the system to be solved, N_{FD}, for the finite-difference case will be $(\eta_\infty/\Delta\eta)$, while the order of the system to be solved by the series method is $4N_s$ where N_s is the number of terms in the series. It is pointed out that the solution for the $4N_s$ quantities using the series method, yields the dependent variable and its higher derivatives whereas the solution of the finite-difference system yields only the dependent variable.

Both methods utilize a linearization technique and the comparisons are given for the converged solution after several iterations; in both cases approximately five iterations of the linearized equation are required for convergence.

The quantity $|\Delta F''_w|$ was chosen as a measure of accuracy where

$$\Delta F''_w = F''_w \text{ (computed)} - F''_w \text{ (exact)}$$

The value of F''_w (exact) is taken from Ref. 8.

Figure 1 shows the relation between accuracy and number of series terms for $\beta = 2.0$, 0.0, and -0.14 and the relation between accuracy and number of difference intervals for $\beta = 0.0$. The correlation line drawn for the series solution is taken to represent the accuracy for the $\beta = 0$ case over the range of $|\Delta F''_w|$ investigated to within an order of magnitude. Both methods utilized double precision arithmetic. It can be seen that for a given accuracy the number of difference intervals required is an order of magnitude larger than the number of series terms. For example, approximately 10 series terms in comparison to 100 difference intervals are required for a value of $|\Delta F''_w|$ of 10^{-4}. Moreover, the minimum value of $|\Delta F''_w|$ attainable with the 5-point central-difference approximation is 1.5×10^{-5} whereas no limit exists for the series solution over the range of $|\Delta F''_w|$ shown.

Both techniques require the solution of a linear system. It can be shown that[11] the solution of a general Nth order system by gauss elimination requires $(N^3/3 + N^2 - N/2)$ operations. Application of the 5-point central-difference operator leads to a set of linear equations with a pentadiagonal band

Table 7 Computed values of shear parameter and displacement thickness for Howarth's retarded flow, $u_e = 1 - x/8$

x	f''_w (present)	f''_w Smith	f''_w Hartree	$\delta^*\left(\frac{\nu x}{u_e}\right)^{1/2}$
0	0.332057	0.332057		
0.2	0.29105			1.8212
0.3	0.26844			1.8806
0.4	0.24407	0.24392	0.24406	1.9483
0.5	0.21754			2.02665
0.6	0.18827			2.1197
0.7	0.15519			2.2348
0.8	0.1162	0.1169	0.1168	2.3867
0.88	0.0769	0.0778	0.0773	2.5636
0.92	0.0507	0.0521	0.0508	2.6989
0.948	0.0226	0.0264	0.0249	2.8703
0.955	0.0109			2.9981
0.958	0.0042[a]	0.0095	0.0059[b]	3.0926
0.9589		0.0065[c]		

[a] Extrapolated to zero at $x = 0.9584$.
[b] Extrapolated to zero at $x = 0.9589$.
[c] Extrapolated to zero at $x = 0.960$.

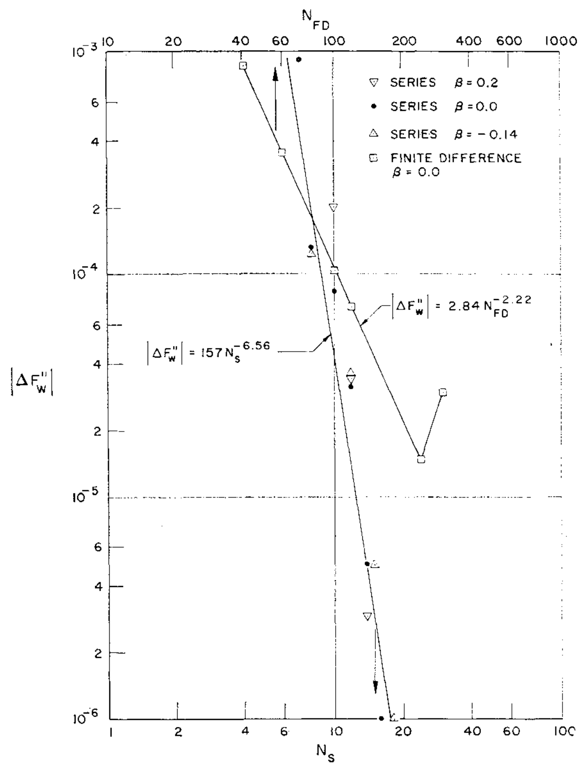

Fig. 1 Accuracy comparison between series and finite-difference methods.

matrix requiring $32N$ operations to solve. The system of order $4N$ to be solved by the series method leads to a sparse operator matrix. Only the first $(N - 2)$ rows are filled by the series coefficients appearing in Eqs. (62a,b). The remaining operations are represented by Eq. (43) for relations between a'_r and a_r, a''_r, and a'_r, and a'''_r and a''_r. These provide submatrixes which are initially triangular and of order N within the $4N$ matrix. The remaining elements are zero. Thus, the only portion of the operator matrix to be triangularized is of the order of the first N rows and N columns. The total number of operations to solve the complete system was found to be of the order of $16N^3/3$; it is pointed out that had the matrix not been partially triangularized and sparse to begin with, the number of operations would have been of the order $(4N)^3/3$. The difficulty in obtaining a solution using the difference technique increases linearly with number of difference intervals whereas it increases as the cube of the number of series terms. However, the accuracy of the difference method increases as the inverse 2.22 power of N_{FD} while the accuracy of the series increases as the inverse 6.56 power of N_s (see Fig. 1). The relative number of operations between the two methods is of the order

$$32\text{N}_{FD}/(16/3)N_s^3$$

Using the correlations shown in Fig. 2, for a given accuracy $\Delta F''_w$ the relative number of computing operations is given by

$$0.95\ |\Delta F''_w|^{0.0069}$$

Therefore over the range of N_{FD} for which the finite-difference solution's accuracy is increasing the ratio of number of operations to attain a given $|\Delta F''_w|$ using the finite-difference method to the number of operations using a series, is of the order of unity.

It is pointed out that the preceding discussion can be used only to provide a crude estimate of the relative computing times between the two methods. Such an estimate depends on the efficiency of the programed logic, a consideration of the relative length of time different operations require (i.e., addi-

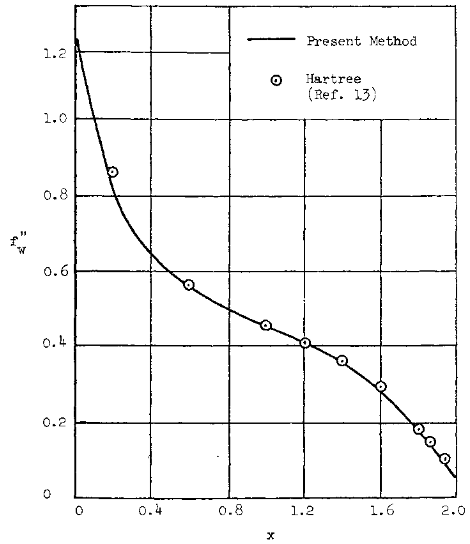

Fig. 2 Streamwise variation of shear parameter for elliptic cylinder.

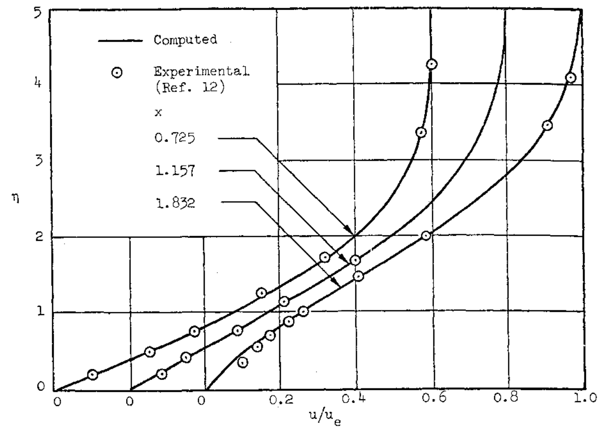

Fig. 3 Comparison of computed and experimental profiles on an elliptic cylinder.

tion vs division), and the type of arithmetic used (fixed or floating point). The latter two considerations will, in turn, depend on the type of computer used. It can be concluded that the accuracy of the finite-difference solution increases at a much lower rate with the order of the algebraic system to be solved while the difficulty of solving the system increases at a greater rate with increasing accuracy using the series solution.

3.4 Comparison of Results with Flow Studied Experimentally

Measurements of a separating boundary layer on an elliptic cylinder were made by Schubauer.[12] This case was computed using the present method and comparisons with Hartree's results and experimental data are given. Twenty-five terms were used and η_∞ was chosen as 8. Figure 2 shows a comparison of the computed values of the shear parameter $f\,'_w$ with values obtained by Hartree[13] and Fig. 3 shows comparisons of experimental and computed velocity profiles. The present results began at $x = 0$ using stagnation point conditions while Hartree began his computation at $x = 1.2$; this could have caused the small differences in results between the two solutions shown in Fig. 2. Separation was observed experimentally at $x = 1.99$. The present method and Hartree's method do not predict separation. Hartree, however, pointed out the sensitivity of the boundary layer to the pressure gradient near separation and indicated that a slight modification, within experimental error, of the measured gradient would lead to an accurate theoretical prediction of separation.

3.5 Concluding Statements

The method presented is capable of providing accurate solutions for similar and nonsimilar flows. Less than ten terms are needed to obtain three or four place accuracy for many flows. Solutions quite close to separation are obtainable. For the comparison made it is shown that the number of arithmetic operations employing the present method are approximately the same as the number using a finite-difference technique for a given accuracy. The series provides the function to be solved for in a closed form that is easily integrable or differentiable and an economical means of representing the solution.

For the cases studied here the number of iterations needed to obtain a solution varied from four to seven and N varied from 8 to 25, while using a value for δ, appearing in Eq. (63) of 10^{-7}.

The application of quasi-linearization made it possible to use an orthogonal set of polynomials to obtain a solution and in this investigation a Chebyshev series was employed. However, use of a linearization technique makes it possible to use other sets and whether or not there are other series that may be better is a problem requiring further study.

References

[1] Smith, A. M. O. and Clutter, D. W., "Solution of the Incompressible Laminar Boundary-Layer Equations," *AIAA Journal*, Vol. 1, No. 9, Sept. 1963, pp. 2062–2071.

[2] Smith, A. M. O. and Clutter, D. W., "Machine Calculation of the Compressible Boundary Layer," *AIAA Journal*, Vol. 3, No. 4, April 1965, pp. 639–647.

[3] Jaffe, N. A., Lind, R. C., and Smith, A. M. O., "Solutions to the Binary Diffusion Laminar Boundary-Layer Equations Including Second-Order Transverse Curvature," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1563–1569.

[4] Hartree, D. R. and Womersley, J. R., "A Method for the Numerical or Mechanical Solution of Certain Types of Partial Differential Equations," *Proceedings of the Royal Society, Ser. A*, Vol. 161, No. 906, Aug. 1937, pp. 353–367.

[5] Bellman, R. E. and Kalaba, R. E., *Quasi-Linearization and Nonlinear Boundary-Value Problems*, American Elsevier, New York, 1965.

[6] Libby, P. A. and Chen, K. K., "Remarks on Quasi-Linearization Applied in Boundary-Layer Calculations," *AIAA Journal*, Vol. 4, No. 5, May 1966, pp. 937–939.

[7] Fox, L., "Chebyshev Methods for Ordinary Differential Equations," *Computer Journal*, Vol. 4, 1962, pp. 318–331.

[8] Smith, A. M. O., "Improved Solutions of the Falkner and Skan Boundary-Layer Equation," Sherman M. Fairchild Paper FF-10, published in Institute of Aeronautical Sciences, New York, 1954.

[9] Hartree, D. R., "Solution of the Laminar Boundary-Layer Equation for Retarded Flow," British ARC R and M 2426, 1949.

[10] Cebeci, T. and Smith, A. M. O., "A Finite-Difference Solution of the Incompressible Turbulent Boundary-Layer Equations by an Eddy-Viscosity Concept," Rept. DAC 67130, Oct. 1968, McDonnell Douglas Aircraft Co.

[11] von Neumann, J. and Goldstine, H. H., "Numerical Inverting of Matrices of High Order," *Bulletin of the American Mathematical Society*, Vol. 53, 1947, pp. 1021–1099.

[12] Schubauer, G. B., "Airflow in a Separation Laminar Boundary Layer," Rept. 527, 1935, NACA.

[13] Hartree, D. R., "The Solution of the Equation of the Laminar Boundary Layer for Schubauer's Observed Pressure Distribution on an Elliptic Cylinder," British R and M 2427, 1949.